Research Statement

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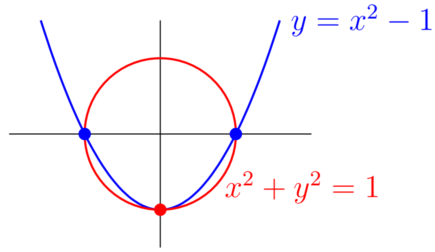
My primary research area is commutative algebra, which is concerned with commutative rings and their modules. Commutative algebra is closely related to algebraic geometry, and many of the projects which I have worked on were inspired by, and have applications to, geometric problems.

For example, Bézout's Theorem states that two plane curves defined by polynomials over a field k of degrees d and e intersect in de points, once a few mild conditions are imposed and a notion of intersection multiplicity is defined.



For example, in the above diagram, there are two curves defined by degree two polynomials. The curves intersect in *simple crossings* at the two blue points, corresponding to a multiplicity of 1, and, but the red point of intersection must be counted with multiplicity 2 to realize Bézout's Theorem in this case, which predicts four interection points. To define the multiplicity of the red point, we use the Hilbert-Samuel multiplicity of the ring $R = k[x, y]/(x^2 + y^2 - 1, x^2 - y - 1)$ localized at the ideal $\mathfrak{m} = (x, y + 1)R$, which corresponds to the red point $(0, -1)$. The multiplicity is the leading coefficient of the Hilbert-Samuel function $HS(n) = \lambda(R/\mathfrak{m}^n)$, where $\lambda(M)$ is the length of the R-module M.

Defining multiplicity, or more specifically Hilbert-Samuel multiplicity, of commutative rings gives us a numerical measure of singularity; that is, the higher the multiplicity of a ring, the more complicated its structure is. Much of my research considers these kinds of numerical invariants.

Several branches of my research are focused on positive characteristic rings. When R is a ring of prime characteristic $p > 0$, then we have several tools available which we don't in characteristic 0. In particular the *Frobenius* function $F : R \to R$ given by $F(x) = x^p$ is a ring homomorphism in this setting, which allows us to consider R as a module over itself in a new way. This gives us new invariants and operations we can describe in positive characteristic, many of them closely analogous to ones in characteristic zero.

My research program can be broadly categorized into three threads:

- Thresholds, including F-pure thresholds and log canonical thresholds, and their relationships.
- Multiplicities, inclding Hilbert-Samuel and Hilbert-Kunz multiplicities, an interpolation between them, and associated closure operations
- Comparing and contrasting various classes of ring self-maps like p^{-e}-linear maps and differential operators and their compatible and fixed ideals.

Thresholds

In an early project of mine, I worked with Linquan Ma, Janet Page, Rebecca R.G., and Wenliang Zhang to take a characteristic 0 theorem of Niu [14] and prove the analogous statement in positive characteristic. The theorem we proved had to do with studying the F-singularities of the generic link of an ideal.

We say that two ideals I and J of a ring R are *linked* if there exists a regular sequence $(a_1, \dots, a_c) \subseteq I$ such that c is the height of I and $J = ((a_1, \dots, a_c) : I)$. Many properties of I imply similar or related properties for J when the ideals are linked. Finding a regular sequence $a_1, \dots, a_c \subseteq I$ can be difficult, so as an alternate approach we can construct such a sequence by passing to a larger ring $S = R[u_{i,j}]$. Here we adjoin an appropriate number of variables in order to take the a_i to be generic combinations of the generators of I; that is, each a_i is a dot product of a row of the matrix $(u_{i,j})$ with the generators of I. Now we can take $J = ((a_1, \dots, a_c) : IS) \subseteq S$, and J is indeed linked to IS. We call J a *generic link* of I.

In [14], Niu shows that the canonical sheaf of S/J, which is an object carrying geometric information about the variety S/J, can be defined in terms of the multiplier ideal $\mathcal{J}(R, I^c)$. As a result he shows that $\mathrm{lct}(S, J) \geq \mathrm{lct}(R, I)$, implying that if (R, I^c) is log canonical, then (S, J^c) is log canonical.

In our paper [8], we proved the following positive characteristic version of Niu's theorem.

Theorem 1. *Suppose that $R = k[x_1, \dots, x_n]$ where k is a field of characteristic $p > 0$, $I \subseteq R$ is a reduced, equidimensional ideal of height c which has a reduction which is a complete intersection, and J is a generic link of I.*

(1) $\tau(\omega_{S/J}) \cong \tau(R, I^c)\cdot(S/J)$, *where $\tau(\omega_{S/J})$ is the parameter test submodule and $\tau(R, I^c)$ is the test ideal of the pair (R, I^c). In parituclar, if $\tau(I^c) = I$ then S/J is F-rational.*
(2) $\mathrm{fpt}(J) \geq \mathrm{fpt}(I)$. *In particular, if (R, I^c) is F-pure, then (S, J^c) is also F-pure.*

Our work on F-singularities of generic links has inspired further research, including [7] and [15]. Some directions for future research in this direction include the following.

Question 1: Can we remove the "has a reduction which is a complete intersection" condition? This condtion is not present in the corresponding characteristic 0 statements from [14], and arises from the fact that many terms in a power of a polynomial may vanish in positive characteristic. Does this mean that we have meaningfully more complexity in positive characteristic, or can we avoid this vanishing using a more clever construction?

Question 2: Linkage has a generalization, *s-residual intersection*, and we can form *generic s-residual intersections* as well. The main difference is that we don't require the regular sequence $a_1, a_2, \dots, a_s$ to have a length equal to the height of I. Which of the results above hold for s-residual intersections?

Multiplicities

For a local ring $(R, \mathfrak{m})$ and an ideal $I \subseteq R$, the Hilbert-Samuel multiplicity of I is defined to be $e(I) = \lim_{n\to\infty} d! \cdot \lambda(R/I^n)/n^d$, where $d = \dim R$ and $\lambda(M)$ is the length of an R-module M. Of particular interest is $e(\mathfrak{m})$, the Hilbert-Samuel multiplicity of the maximal ideal itself, which we also denote by $e(R)$. This value, which is always a positive integer, carries information about the structure of the ring. For instance, if R is a regular ring, then $e(R) = 1$. A classical theorem of Nagata proves that if R is equidimensional, the converse holds as well.

If a ring R is of positive prime characteristic p, and $I \subseteq R$ is an ideal, then the ideal generated by elements of the form x^{p^e} for $x \in I$ is denoted by $I^{[p^e]}$ and is called the eth Frobenius power of I. The Frobenius power $I^{[p^e]}$ is contained in I^{p^e} and is usually much smaller. The Hilbert-Kunz multiplicity of I is $e_{HK}(I) = \lim_{e\to\infty} \lambda(R/I^{[p^e]})/p^{ed}$. Like the Hilbert-Samuel multiplicity, the Hilbert-Kunz multiplicity is a real number at least 1, though unlike the Hilbert-Samuel multiplicity it can be non-integer, and even irrational. The Hilbert-Kunz multiplicity is a more sensitive measure of singularity than the Hilbert-Samuel multiplicity. Many theorems true of Hilbert-Samuel multiplicity have analogous versions for Hilbert-Kunz multiplicity, though the Hilbert-Kunz versions are often more difficult to prove.

Interpolating Between Multiplicities. To relate Hilbert-Samuel and Hilbert-Kunz multiplicities, I investigated a function that interpolates between the Hilbert-Samuel multiplicity of I and the Hilbert-Kunz multiplicity of J as a positive real parameter s varies, for two ideals I and J of a ring R. The function is defined by

$$e_s(I,J) = \lim_{e\to\infty} \frac{\lambda\left(R/(I^{\lceil sp^e \rceil} + J^{[p^e]})\right)}{p^{ed}\mathcal{H}_s(d)}$$

where $\mathcal{H}_s(d)$ is a normalizing function which guarantees that if R is a regular ring, then $e_s(\mathfrak{m},\mathfrak{m}) = 1$ for all s. In [20] I proved that this limit always exists for $\mathfrak{m}$-primary ideals I and J, and established that for each s, this function satisfies many of the properties that Hilbert-Samuel and Hilbert-Kunz multiplicities. This function does indeed interpolate between the two multiplicities in a way that is related two two other limits in positive characteristic. In particular, $e_s(I,J) = e_{\mathrm{HK}}(J)$ whenever s is greater than both the dimension of the ring and the F-threshold of I with respect to J, a value defined in [13] and proved to exist in [3]. On the other end of the spectrum, $e_s(I,J) = e(I)$ whenever s is less than both 1 and a threshold based on a dual notion to that of the F-threshold. As a function of s, $e_s(I,J)$ is continuous.

Understanding the mixed powers $I^{\lceil sp^e \rceil} + J^{[p^e]}$ which appear in the definition of the s-multiplicity is key to proving results about the s-multiplicity and related ideas. This problem can be attacked combinatorially, which is how I established certain upper bounds for the s-multiplicity and proved

As part of my work I was able to provide a method for computing the s-multiplicity of a pair of monomial ideals in toric space by expressing it as a volume in d-dimensional real space. This construction, being very visual, helped to build intuition and understanding of the nature of the s-multiplicity function.

I collaborated with Lance E. Miller to write two papers related to s-multiplicity. In [11], we built upon several arguments of Watanabe and Yoshida to establish bounds for s-multiplicity in terms of the Hilbert-Samuel and Hilbert-Kunz multiplicities. For instance, we showed that $e_s(I,I)$ is constant in s if and only if $e(I,I) = e_{HK}(I,I)$. If R is Cohen-Macaulay, then we recover a significantly generalized version of a lower bound established in [22]. In particular, if J is a parameter ideal reduction of I, then for any $1 \le t \le s$, we have that

$$e_s(I) \ge \left(\frac{\mathcal{H}_t(d) - \mu(I/J^*)\mathcal{H}_{t-1}(d)}{\mathcal{H}_s(d)}\right) e(I).$$

We also examined an s-multiplicity version of a famous conjecture of Watanabe and Yoshida concerning the minimal values of $e_{HK}(R)$ for singular rings. In particular, we considered the following question: If R is a singular ring of dimension d, is it the case that $e_s(R) \ge e_s(R_d)$, where $R_d = k[[x_0,\ldots,x_d]]/(\sum_i x_i^2)$? We were able to establish a positive answer to this question in the Cohen-Macaulay case in dimension 3 or less and in the complete intersection case when $p \geq 3$.

In [10], we examined the values $\ell(p^e) = \lambda(R/(\mathfrak{m}^{\lceil sp^e \rceil} + \mathfrak{m}^{[p^e]}))$ for $R = k[X]/I_2(X)$, where X is a matrix of indeterminates and $I_2(X)$ is the ideal generated by all 2×2 minors of X. We were expanding upon the results appearing in [9] and [17] on the Hilbert-Kunz function. Using Gröbner bases and combinatorial arguments, we derived a closed form for $\ell(p^e)$. Two consequences of this result are a way of calculating the s-multiplicity of 2×2 determinantal rings by examining the leading term of this closed form, and a proof that if $s \in \mathbb{Z}[p^{-1}]$, then $\ell(p^e)$ is eventually polynomial in p^e. This second result is notable since in general the Hilbert-Samuel function $\lambda(R/\mathfrak{m}^n)$ is a polynomial in n but the Hilbert-Kunz function $\lambda(R/\mathfrak{m}^{[p^e]})$ is not a polynomial in p^e.

s-Closures. Hilbert-Samuel multiplicity is closely related to integral closure of ideals, and Hilbert-Kunz multiplicity is closely related to tight closure of ideals, so it was natural to search for a family of closure operators that would be related in the same way to s-multiplicity. I began studying these closures in [20] and continued in [21].

The most straightforward definition of an operator related to the s-multiplicity is to define that $x \in I^{\{s\}}$ if there exists some c not in any minimal prime of R such that for all $e \gg 0$, we have that $cx^{p^e} \in I^{\lceil sp^e \rceil} + I^{[p^e]}$. We call this operation *weak s-closure*. Weak 1-closure is integral closure, and for large s, weak s-closure is tight closure. However, for intermediate values of s it is not obvious that this operation is *idempotent*, i.e. it isn't clear that $(I^{\{s\}})^{\{s\}} = I^{\{s\}}$. AS we work in noetherian rings, we can define a true closure operation by defining the *s-closure* of I to be the ideal we obtain my applying the weak s-closure repeatedly until the ideal stabilizes. We denote the s-closure of I by I^{cl_s}.

In [20], I proved that if $x \in I^{\mathrm{cl}_s}$, then $e_s((I,x),(I,x)) = e_s(I,I)$, recovering the analogous statement for Hilbert-Samuel (resp. Hilbert-Kunz) multiplicity and integral (resp. tight) closure. Also interesting is the converse direction: if $J \subseteq I$ and $e_s(J,J) = e_s(I,I)$, does $J^{\mathrm{cl}_s} = I^{\mathrm{cl}_s}$? I was able to prove this via a theorem of Polstra and Tucker in [16] in the case that the ring R is an F-finite complete domain. In [21], I extended this to the F-finite complete unmixed case by proving that membership in the s-closure can be checked modulo minimal primes and using the Associativity Formula for s-multiplicity from my previous paper:

$$e_s(I,J) = \sum_{\mathfrak{p} \in \operatorname{Assh} R} e_s(I(R/\mathfrak{p}), J(R/\mathfrak{p}))\lambda_{R_\mathfrak{p}}(R_\mathfrak{p}),$$

where Assh R denotes the set of prime ideals $\mathfrak{p} \subseteq R$ such that $\dim R/\mathfrak{p} = \dim R$.

Also in [21], I established several important results regarding the structure of s-closure. In the case of graded rings, and inspired by work of Smith in [19], I showed that the s-closure of a graded ideal is graded and in certain cases gave necessary and sufficient degree conditions for an element to be contained in the s-closure. I also showed that several common conditions on ideals guarantee that the weak s-closure is the s-closure. For instance, if I is principal, a monomial ideal, or a power of the ideal of positively graded elements in an $\mathbb{N}$-graded ring, then $I^{\{s\}} = I^{\mathrm{cl}_s}$ for all s.

I also generalized the Hochster-Huneke version of the Brianọn-Skoda Theorem in [4] relating integral and tight closure to the case of two s-closures.

Theorem 2. *Let R be a ring, $1 \leq t < s$, and I an ideal of R. If $r \geq \frac{(\mu(I)-1)(s-t)}{t(s-1)}$, then for all $n \in \mathbb{N}$, $(I^{n+r})^{\{t\}} \subseteq (I^n)^{\{s\}}$.*

The well-known Brianȍn-Skoda Theorem corresponds to the case $t = 1$, $s = d$.

Natural questions for future research regarding s-multiplicity and s-closures include the following.

Question 1: Can we use the fact that Lech's Conjecture holds for Hilbert-Kunz multiplicity and the continuously interpolating nature of $e_s(I, J)$ to conclude that Lech's Conjecture holds in more cases? As a first step in that direction, can we reduce the conjecture to a more tractable problem based on the interpolation? Lech's Conjecture is one of the longest standing open problems in commutative algebra and even partial results are interesting.

Question 2: If $e_s(\mathfrak{m}, \mathfrak{m}) = 1$ for some $s \geq 0$, is the ring regular? This question is based on the fact that for a reasonably nice ring (equidimensional is sufficient), if either the Hilbert-Samuel or the Hilbert-Kunz multiplicities are equal to 1, then the ring is regular. I would like to know if it suffices for the s-multiplicity to be 1 for any s.

Question 3: Is there a natural way to extend the notion of s-multiplicity to non-$\mathfrak{m}$-primary ideals? There have been several explorations of how to define Hilbert-Samuel and Hilbert-Kunz multiplicities for non-$\mathfrak{m}$-primary ideals. If I could find a reasonable, natural way to extend s-multiplicity to those cases as well, it could provide a unifying context to work in. A possible first step is to examine what happens when only one of I and J are $\mathfrak{m}$-primary. The existence of the limit defining $e_s(I, J)$ can still be shown in broad cases using a theorem of Polstra and Tucker in [16], so this may provide some insight.

Differential Operators

If R is a commutative ring with prime characteristic $p > 0$, then there are several classes of F-singularities we define. This work began with a celebrated theorem of Kunz ([6]) which shows that R is a regular ring if and only if R is a free R-module under the Frobenius action defined by $r \cdot x = r^p x$. Weakening the condition of being a free module gives the various F-singularities. For instance, R is called *F-pure* if R contains at least one copy of itself as a module under the Frobenius action, and R is called *strongly F-regular* if

Given a commutative ring R with characteristic $p > 0$, a p^{-e}-linear map is an additive function $\varphi : R \to R$ such that $\varphi(x^p y) = x\varphi(y)$ for all $x, y \in R$. These maps are used to define several singularities in positive characteristic. For example, a ring R is called *strongly F-regular* if, for every $x \in R$, if x is not in a minimal prime of R, then there exists $e \geq 0$ and a p^{-e}-linear map φ such that $\varphi(x) = 1$. An equivalent way to express this condition is that there is no nonzero proper ideal I of R such that $\varphi(I) \subseteq I$ for all p^{-e}-linear maps φ. This gives us motivation to study ideals fixed by p^{-e}-linear maps.

Ideals Fixed by Maps. In [5], the authors study this question for toric ideals. For our purposes, we may treat these as monmomial ideals in k-algebras generated by a set of monomials. In [5], the authors show that in this setting, for a p^{-e}-linear map φ, there are a nonempty finite collection of ideals I that are φ-fixed. Furthermore, these ideals have a very special structure; most monomial ideals are not fixed by any p^{-e}-linear map.

Recent results in [2] showed that differential operators can be used to define signatures, i.e. numerical measures of singularities, just as p^{-e}-linear maps can. This inspired Lance Edward Miller, Janet Vassilev, and I to ask which toric ideals are fixed by differential operators. We were able to prove the following theorem, which was unexpected given the analogy we were inspired by. In this theorem we use a description of the differential operators on a complex toric ring given by Saito and Traves in [18]. Here δ is a differential operator of multidegree $\mathbf{d}$, and $f \in \mathbb{C}[x_1, \ldots, x_d]$ is a polynomial such that for any monomial $x^{\mathbf{a}}$, we have that

$\delta(x^{\mathbf{a}}) = f(\mathbf{a}) \cdot x^{\mathbf{a}+\mathbf{d}}$. The set $V_{\mathrm{mon}}(f)$ is the set of lattice points $\mathbf{a}$ with $f(\mathbf{a}) = 0$, and for any lattice point $\mathbf{a}$, $\nu(\mathbf{a})$ is a function which essentially measures how far you have to travel starting at $\mathbf{a}$ to reach the first point in $V_{\mathrm{mon}}(f)$ which lies on the line parallel to the vector $\mathbf{d}$. The function ν' is a shift of ν for technical reasons.

Theorem 3. [12] *Let $R = \mathbb{C}[S]$ be an affine semigroup ring defined by a cone $\sigma^\vee$, $I \subseteq R$ a proper monomial ideal, and $\mathbf{d} \in \mathbb{Z}^d$ such that $-\mathbf{d} \in S$. Set $\mathbf{e} \in \mathbb{Z}^d$ primitive such that $\mathbf{d} = q\mathbf{e}$ for some $q \in \mathbb{N}$.*

- *There exists a homogeneous differential operator δ of degree $\mathbf{d}$ such that I is δ-fixed.*
- *If $-\mathbf{d} \in \mathrm{int}\sigma^\vee$, then δ fixes a monomial ideal I if and only if*
  (1) *$\nu(\mathbf{a}) > -\infty$ for all $\mathbf{a} \in S$,*
  (2) *for all $\mathbf{a} \in V'_{\mathrm{mon}}(f)$ and $i = 0, \ldots, q-1$, $\mathbf{a} - i\mathbf{e} \in V_{\mathrm{mon}}(f)$, and*
  (3) *for every $\mathbf{a}, \mathbf{b} \in V'_{\mathrm{mon}}(f)$, $\mathbf{a} - \mathbf{b} \notin S - \mathbf{e}$.*
  In this case, δ fixes only one ideal I with $\mathrm{Exp}I = \{\mathbf{a} \in S \mid \nu'(\mathbf{a}) \geq 0\}$.

The theorem above is constructive and quite explicit. We observed that in contrast to the p^{-e}-linear case, most differential operators don't fix any ideals, and those that do will typically fix only one. On the other hand, and even more unexpected, we find that every monomial ideal is fixed by a family of differential operators.

The theorem shows that the data of an ideal, such as whether it is the unit ideal, is determined by one of the differential operators fixing it. This opens up the possibility to detect properties of an ideal by examining a differential operator. We developed an example to demonstrate this technique by showing that we can calculate the log canonical threshold of an ideal by constructing a family of differential operators and measuring which of these differential operators have the monomial 1 in their image. We approched this by first providing a general construction which gives a differential operator which fixes an ideal defined by a polyhedron or union of polyhedra. Specifically, if P is a polyhedron in the cone $\sigma^\vee$, and I_P is the ideal whose exponent set is the set of lattice points in P, then we can construct a differential operator δ fixing I. By constructing a family of polytopes P_c corresponding to the multiplier ideals of pairs (R, I^c), we have a family of differential operators δ_c fixing the multiplier ideals. We can then use δ_c to detect which of these multiplier ideals are proper, and the least such c is the log canonical threshold of I.

There are several questions which naturally arise from this line of research.

Question 1: How can we develop a similar theory in positive characteristic? The Saito-Traves description of differential operators only holds over the complex numbers, and the structure of the differential operators in positive characteristic is more complicated. Even computing families of examples in polynomial rings over finite fields would be helpful in developing the theory.

Question 2: Given a monomial ideal I, what is the least order of a differential operator which fixes I? This is related to the degree of the polynomial defining a given differential operator. Providing such bounds will be important for making any computational approach towards this problem efficient.

Question 3: What can we say about the non-normal case? All the toric rings we have studied so far have been normal, but differential operators over non-normal semigroup rings have been described and studied in [1], so it should be possible to extend our results to that setting.

Question 4: Can we describe ideals fixed by non-homogenous differential operators?

Question 5: Can we describe a general theory of self-maps which includes differential operators, p^{-e}-linear maps, and similar, and prove more general results? Lance Miller and I have already begun discussing this question in some depth and are working on a project which will make progress toward answering this question.

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